3.29. ITERATIVE METHODS

So far we have studied some direct methods which yield the solution after a certain amount of fixed computation, for the solution of simultaneous linear equations. Now we shall discuss the iterative or indirect methods. In these methods we start from an approximation to the true solution and, if convergent, derive a sequence of closer approximations. We repeat the cycle of computations till the required accuracy is obtained.

But the method of iteration is not applicable to all systems of equations. For this, in the system each equation of the system must contain one large coefficient (much larger than the others in that equation) and the large co-efficient must be attached to a different unknown in that equation.

In other words the solution of a system of linear equations will exist by iterative procedure **if the absolute value of the largest co-efficient is greater than the sum of the absolute values of all remaining co-efficients in each equation** (condition for convergence).

Now we shall go through some of such methods.

3.29.1. Jacobi Method of Iteration or Gauss-Jacobi Method

Consider the system of equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad \text{...(1)}$$

Let $|a_1| > |b_1| + |c_1|$; $|b_2| > |a_2| + |c_2|$; $|c_3| > |a_3| + |b_3|$

i.e., in each equation the co-efficients of the diagonal terms are large. Hence the system (1) is ready for iteration. Solving for x, y, z respectively, we get

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}\right\} \quad \text{...(2)}$$

Let x_0, y_0, z_0 be the initial approximations of the unknowns x, y, z. Substituting these on R.H.S. of (2) the first approximations are given by

$$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$

$$y_1 = \frac{1}{b_2}(d_2 - a_2x_0 - c_2z_0)$$

$$z_1 = \frac{1}{c_3}(d_3 - a_3x_0 - b_3y_0)$$

Substituting the values x_1, y_1, z_1 in the R.H.S. of (2), the second approximations are given by

$$x_2 = \frac{1}{a_1}(d_1 - b_1y_1 - c_1z_1)$$

$$y_2 = \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_1)$$

$$z_2 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$$

Proceeding in the same way if x_r, y_r, z_r are the rth iterates then

$$\mathbf{x_{r+1}} = \frac{1}{\mathbf{a_1}}(\mathbf{d_1} - \mathbf{b_1y_r} - \mathbf{c_1z_r})$$

$$\mathbf{y_{r+1}} = \frac{1}{\mathbf{b_2}}(\mathbf{d_2} - \mathbf{a_2x_r} - \mathbf{c_2z_r})$$

$$\mathbf{z_{r+1}} = \frac{1}{\mathbf{c_3}}(\mathbf{d_3} - \mathbf{a_3x_r} - \mathbf{b_3y_r})$$

The process is continued till convergency is secured.

Note. In the absence of any better estimates the initial approximations are taken as $x_0 = 0$, $y_0 = 0, z_0 = 0$.

Example. *Solve by Jacobi iteration method the system $8x - 3y + 2z = 20$; $6x + 3y + 12z = 35$; and $4x + 11y - z = 33$.*

Sol. Consider the given system as

$$8x - 3y + 2z = 20$$
$$4x - 11y - z = 33$$
$$6x + 3y + 12z = 35$$

so that the diagonal elements are dominant in the co-efficient matrix. Now we write the equations in the form

$$\left.\begin{aligned} x &= \frac{1}{8}(20 + 3y - 2z) \\ y &= \frac{1}{11}(33 - 4x + z) \\ z &= \frac{1}{12}(35 - 6x - 3y) \end{aligned}\right\} \qquad \ldots(1)$$

We start from an approximation $x_0 = y_0 = z_0 = 0$ substituting these on R.H.S. of (1), we get

First approximation

$$x_1 = \frac{1}{8}[20 + 3(0) - 2(0)] = 2.5$$
$$y_1 = \frac{1}{11}[33 - 4(0) + 0] = 3$$
$$z_1 = \frac{1}{12}[35 - 6(0) - 3(0)] = 2.9166667$$

Second approximation

Substituting x_1, y_1, z_1 on R.H.S. of (1), we get

$$x_2 = \frac{1}{8}[20 + 3(3) - 2(2.9166667)] = 2.8958333$$
$$y_2 = \frac{1}{11}[33 - 4(2.5) + 2.9166667] = 2.3560606$$
$$z_2 = \frac{1}{12}[35 - 6(2.5) - 3(3)] = 0.9166666$$

Third approximation

Substituting x_2, y_2, z_2 on R.H.S. of (1), we get

$$x_3 = \frac{1}{8}[20 + 3(2.3560606) - 2(0.9166666)] = 3.1543561$$
$$y_3 = \frac{1}{11}[33 - 4(2.8958333) + 0.9166666] = 2.030303$$
$$z_3 = \frac{1}{12}[35 - 6(2.8958333) - 3(2.3560606)] = 0.8797348$$

Fourth approximation

Substituting x_3, y_3, z_3 on R.H.S. of (1), we get

$$x_4 = \frac{1}{8}[20 + 3(2.030303) - 2(0.8797348)] = 3.0414299$$
$$y_4 = \frac{1}{11}[33 - 4(3.1543561) + 0.8797348] = 1.9329373$$
$$z_4 = \frac{1}{12}[35 - 6(3.1543561) - 3(2.030303)] = 0.8319128$$

Fifth approximation

Substituting x_4, y_4, z_4 on R.H.S. of (1), we get

$$x_5 = \frac{1}{8}[20 + 3(1.9329373) - 2(0.8319128)] = 3.0168733$$

$$y_5 = \frac{1}{11}[33 - 4(3.0414299) + 0.8319128] = 1.9696539$$

$$z_5 = \frac{1}{12}[35 - 6(3.0454299) - 3(1.9329373)] = 0.9127173$$

Sixth approximation

Substituting x_5, y_5, z_5 on R.H.S. of (1), we get

$$x_6 = \frac{1}{8}[20 + 3(1.9696539) - 2(0.9127173)] = 3.0104409$$

$$y_6 = \frac{1}{11}[33 - 4(3.0168733) + 0.9127173] = 1.9859295$$

$$z_6 = \frac{1}{12}[35 - 6(3.0168733) - 3(1.9696539)] = 0.9158165$$

Seventh approximation

Substituting x_6, y_6, z_6 on R.H.S. on (1), we get

$$x_7 = \frac{1}{8}[20 + 3(1.9859295) - 2(0.9158165) = 3.0157694$$

$$y_7 = \frac{1}{11}[33 - 4(3.0104409) + 0.9158165] = 1.9885503$$

$$z_7 = \frac{1}{12}[35 - 6(3.0104409) - 3(1.9859295)] = 0.9149638$$

Eight approximation

Substituting x_7, y_7, z_7 on R.H.S. of (1), we get

$$x_8 = \frac{1}{8}[20 + 3(1.9885503) - 2(0.9149638)] = 3.0169654$$

$$y_8 = \frac{1}{11}[33 - 4(3.0157694) + 0.9149638] = 1.9865351$$

$$z_8 = \frac{1}{12}[35 - 6(3.0157694) - 3(1.9885503)] = 0.9116443$$

Ninth approximation

Substituting x_8, y_8, z_8 on R.H.S. of (1), we get

$$x_9 = \frac{1}{8}[20 + 3(1.9865351) - 2(0.9116443)] = 3.0170396$$

$$y_9 = \frac{1}{11}[33 - 4(3.0169654) + 0.9116443] = 1.9857984$$

$$z_9 = \frac{1}{12}[35 - 6(3.0169654) - 3(1.9865351)] = 0.9115501$$

Tenth approximation

Substituting x_9, y_9, z_9 on R.H.S. of (1), we get

$$x_{10} = \frac{1}{8}[20 + 3(1.9857984) - 2(0.9115501)] = 3.0167869$$

$$y_{10} = \frac{1}{11}[33 - 4(3.0170396) + 0.9115501] = 1.9857629$$

$$z_{10} = \frac{1}{12}[35 - 6(3.0170396) - 3(1.9857984)] = 0.9116972$$

Eleventh approximation

Substituting x_{10}, y_{10}, z_{10} on R.H.S. of (1), we get

$$x_{11} = \frac{1}{8}[20 + 3(1.9857629) - 2(0.9116972)] = 3.0167368$$

$$y_{11} = \frac{1}{11}[33 - 4(3.0167869) + 0.9116972] = 1.9858681$$

$$z_{11} = \frac{1}{12}[35 - 6(3.0167869) - 3(1.9857629)] = 0.9118326$$

Twelfth approximation

Substituting x_{11}, y_{11}, z_{11} on R.H.S. of (1), we get

$$x_{12} = \frac{1}{8}[20 + 3(1.9858681) - 2(0.9118326)] = 3.0167424$$

$$y_{12} = \frac{1}{11}[33 - 4(3.0167368) + 0.9118326] = 1.9858987$$

$$z_{12} = \frac{1}{12}[35 - 6(3.0167368) - 3(1.9858681)] = 0.9118312$$

$\therefore$ From the 11th and 12th approximations the values of x, y, z are same correct to four decimal places. Stopping at this stage, we get $x = 3.0167, y = 1.9858, z = 0.9118$.

3.29.2. Gauss-Seidel Iteration Method

This is a modification of Gauss-Jacobi method. As before the system of the linear equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3 \text{ is written as}$$

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}\right\} \quad \text{...(1)}$$

and we start with the initial approximation x_0, y_0, z_0. Substituting y_0 and z_0 in the first equation of (1), we get

$$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$

Now substituting $x = x_1, z = z_0$ in the second equation of (1), we get

$$y_1 = \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_0)$$

Substituting $x = x_1, y = y_1$ in third equation of (1), we get

$$z_1 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$$

This process is continued till the values of x, y, z are obtained to desired degree of accuracy. The general algorithm is as follows :

If x_k, y_k, z_k are the kth iterates then

$$\mathbf{x_{k+1}} = \frac{1}{a_1}(d_1 - b_1y_k - c_1z_k)$$

$$\mathbf{y_{k+1}} = \frac{1}{b_2}(d_2 - a_2x_{k+1} - c_2z_k)$$

and

$$\mathbf{z_{k+1}} = \frac{1}{c_3}(d_3 - a_3x_{k+1} - b_3y_{k+1}).$$

Since the current values of the unknowns at each stage of iteration are used in proceeding to next stage of iteration, this method is more rapid in convergence than Gauss-Jacobi method.

The rate of convergence of Gauss-Seidel method is roughly twice to that of Gauss-Jacobi and the condition of convergence is same as we stated earlier.

Note. Gauss Seidel iteration method converges only for special system of equations. In general the round of errors will be small in iteration methods more ever these are self-correcting methods. Any error made in computation, will be corrected in the subsequent iterations.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the equations by Gauss-Seidel iteration method.*

$$8x - 3y + 2z = 20$$
$$4x + 11y - z = 33$$
$$6x + 3y + 12z = 35$$

Sol. From the given equations, we have

$$x = \frac{1}{8}(20 + 3y - 2z) \quad ...(1)$$

$$y = \frac{1}{11}(33 - 4x + z) \quad ...(2)$$

$$z = \frac{1}{12}(35 - 6x - 3y) \quad ...(3)$$

Putting $y = 0, z = 0$ in the R.H.S. of (1), we get

$$x_1 = \frac{20}{8} = 2.5$$

Putting $x = 2.5, z = 0$ in the R.H.S. of (2), we get

$$y_1 = \frac{1}{11}[33 - 4(2.5)] = 2.0909091$$

Putting $x = 2.5, y = 2.0909091$ in R.H.S. of (3), we get

$$z_1 = \frac{1}{11}[35 - 6(2.5) - 3(2.0909091)] = 1.1439394$$

For the second approximation

$$x_2 = \frac{1}{8}[20 + 3y_1 - 2z_1] = \frac{1}{8}[20 + 3(2.0909091) - 2(1.1439394)] = 2.9981061$$

$$y_2 = \frac{1}{11}[33 - 4x_2 + z_1] = \frac{1}{11}[33 - 4(2.9981061) + 1.1439394] = 2.0137741$$

$$z_2 = \frac{1}{12}[35 - 6x_2 - 3y_2] = \frac{1}{12}[35 - 6(2.9981061) - 3(2.0137741)] = 0.9141701$$

Third approximation

$$x_3 = \frac{1}{8}[20 + 3(2.0137741) - 2(0.9141701)] = 3.0266228$$

$$y_3 = \frac{1}{11}[33 - 3(3.0266228) + 0.9141701] = 1.9825163$$

$$z_3 = \frac{1}{12}[35 - 6(3.0266228) - 3(1.9825563)] = 0.9077262$$

Fourth approximation

$$x_4 = \frac{1}{8}[20 + 3(1.9825163) - 2(0.9077262)] = 3.0165121$$

$$y_4 = \frac{1}{11}[33 - 4(3.0165121) + 0.9077262] = 1.9856071$$

$$z_4 = \frac{1}{12}[35 - 6(3.0165121) - 3(1.9856071)] = 0.9120088$$

Fifth approximation

$$x_5 = \frac{1}{8}[20 + 3(1.9856071) - 2(0.9120088)] = 3.0166005$$

$$y_5 = \frac{1}{11}[33 - 4(3.0166005) + 0.9120088] = 1.9859643$$

$$z_5 = \frac{1}{12}[35 - 6(3.0166005) - 3(1.9859643)] = 0.9118753$$

Sixth approximation

$$x_6 = \frac{1}{8}[20 + 3(1.9859643) - 2(0.9118753)] = 3.0167678$$

$$y_6 = \frac{1}{11}[33 - 4(3.0167678) + 0.9118753] = 1.9858913$$

$$z_6 = \frac{1}{12}[35 - 6(3.0167678) - 3(1.9858913)] = 0.9118099$$

Seventh approximation

$$x_7 = \frac{1}{8}[20 + 3(1.9858913) - 2(0.9118099)] = 0.0167568$$

$$y_7 = \frac{1}{11}[33 - 4(3.0167568) + 0.9118099] = 1.9858894$$

$$z_7 = \frac{1}{12}[35 - 6(3.0167568) - 3(1.9858894)] = 0.9118159.$$

Since at the sixth and seventh approximations the values of x, y, z are same correct to four decimal places, we can stop the iteration process.

$\therefore \quad x = 3.0167, y = 1.9858, z = 0.9118$

We find that 12 iterations are necessary in Gauss-Jacobi method to get the same accuracy as achieved by 7 iterations in Gauss-Seidel method.

Example 2. *Using Gauss-Seidel iteration method solve the system of equations*

$$10x - 2y - z - w = 3\ ;\ -2x + 10y - z - w = 15\ ;$$
$$-x - y + 10z - 2w = 27\ ;\ -x - y - 2z + 10w = -9.$$

Sol. The co-efficient matrix of the given system is diagonally dominant. Hence we can apply Gauss-Seidel iteration method.

From the given equations, we can write

$$x = \frac{1}{10}[3 + 2y + z + w] \quad ...(1)$$

$$y = \frac{1}{10}[15 + 2x + z + w] \quad ...(2)$$

$$z = \frac{1}{10}[27 + x + y + 2w] \quad ...(3)$$

$$w = \frac{1}{10}[-9 + x + y + 2z] \quad ...(4)$$

First approximation

Putting $y = z = w = 0$ R.H.S. of (1), we get

$$x_1 = \frac{3}{10} = 0.3$$

Putting $x = 0.3, z = w = 0$ on R.H.S. of (2), we get

$$y_1 = \frac{1}{10}[15 + 2(0.3)] = 1.56$$

Putting $x = 0.3, y = 1.56, w = 0$ on R.H.S. of (3), we get

$$z_1 = \frac{1}{10}[27 + 0.3 + 1.56] = 2.886$$

Putting $x = 0.3, y = 1.56, z = 2.886$ on R.H.S. of (4), we get

$$w_1 = \frac{1}{10}[-9 + 0.3 + 1.56 + 2(2.886)] = -0.1368$$

Second approximation

$$x_2 = \frac{1}{10}[3 + 2(1.56) + 2.886 - 0.1368] = 0.88692$$

$$y_2 = \frac{1}{10}[15 + 2(0.88692) + 2.886 - 0.1368] = 1.952304$$

$$z_2 = \frac{1}{10}[27 + 0.88692 + 1.952304 + 2(-0.1368)] = 2.9565624$$

$$w_2 = \frac{1}{10}[-9 + 0.88692 + 1.952304 + 2(2.9565624)] = 0.0247651$$

Third approximation

$$x_3 = \frac{1}{10}[3 + 2(1.952304) + 2.9565624 - 0.0247651] = 0.9836405$$

$$y_3 = \frac{1}{10}[15 + 2(0.9836405) + 2.9565624 - 0.0247651] = 1.9899087$$

$$z_3 = \frac{1}{10}[27 + 0.9836405 + 1.9899087 + 2(-0.0247651)] = 2.9924019$$

$$w_3 = \frac{1}{10}[-9 + 0.9836405 + 1.9899087 + 2(2.9924019)] = -0.0041647$$

Fourth approximation

$$x_4 = \frac{1}{10}[3 + 2(1.9899087) + 2.9924019 - 0.0041647] = 0.9968054$$

$$y_4 = \frac{1}{10}[15 + 2(0.9968054) + 2.9924019 - 0.0041647] = 1.9981848$$

$$z_4 = \frac{1}{10}[27 + 0.9968054 + 1.9981848 + 2(-0.0041647)] = 2.9986661$$

$$w_4 = \frac{1}{10}[-9 + 0.9968054 + 1.9981848 + 2(2.9986661)] = -0.0007677$$

Fifth approximation

$$x_5 = \frac{1}{10}[3 + 2(1.9981848) + 2.9986661 - 0.0007677] = 0.9994268$$

$$y_5 = \frac{1}{10}[15 + 2(0.9994268) + 2.9986661 - 0.0007677] = 1.9996752$$

$$z_5 = \frac{1}{10}[27 + 0.9994268 + 1.9996752 + 2(-0.0007677)] = 2.9997567$$

$$w_5 = \frac{1}{10}[-9 + 0.9994268 + 1.9996752 + 2(2.9997567)] = -0.0001384$$

Sixth approximation

$$x_6 = \frac{1}{10}[3 + 2(1.9996752) + 2.9997567 - 0.0001384] = 0.9998968$$

$$y_6 = \frac{1}{10}[15 + 2(0.9998968) + 2.9997567 - 0.0001384] = 1.9999412$$

$$z_6 = \frac{1}{10}[27 + 0.9998968 + 1.9999412 + 2(-0.0001384)] = 2.9999561$$

$$w_6 = \frac{1}{10}[-9 + 0.9998968 + 1.9999412 + 2(2.9999561)] = -0.0002498$$

Seventh iteration

$$x_7 = \frac{1}{10}[3 + 2(1.9999412) + 2.9999561 - 0.0002498] = 0.9999588$$

$$y_7 = \frac{1}{10}[15 + 2(0.9999588) + 2.9999561 - 0.0002498] = 1.9999624$$

$$z_7 = \frac{1}{10}[27 + 0.9999588 + 1.9999624 + 2(-0.0002498)] = 2.9999422$$

$$w_7 = \frac{1}{10}[-9 + 0.9999588 + 1.9999624 + 2(2.9999422)] = -0.0001945$$

Now from sixth and seventh approximations the values of x, y, z correct to four decimal places are

$$x = 0.9999,\ y = 1.9999,\ z = 2.9999,\ w = -0.0002.$$

3.29.3. Relaxation Method

Consider the equations $a_1x + b_1y + c_1z = d_1$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

We define the residuals r_1, r_2, r_3 by the relations

$$\left.\begin{aligned} r_1 &= d_1 - a_1x - b_1y - c_1z \\ r_2 &= d_2 - a_2x - b_2y - c_2z \\ r_3 &= d_3 - a_3x - b_3y - c_3z \end{aligned}\right\} \quad ...(1)$$

To start with, we assume $x = y = z = 0$ and calculate the initial residuals. Then these residuals are reduced step by step by giving increments to the variables. If we can find x, y, z such that the residuals $r_1 = r_2 = r_3 = 0$ then those values of x, y, z are the exact values. Otherwise we liquidate the residuals smaller and smaller and finally negligible to get better approximate values of x, y, z. For this purpose we construct an **operation table** as shown below :

x	y	z	r_1	r_2	r_3
1	0	0	$-a_1$	$-a_2$	$-a_3$
0	1	0	$-b_1$	$-b_2$	$-b_3$
0	0	1	$-c_1$	$-c_2$	$-c_3$

From equations (1) we can see that if x is increased by 1, keeping y and z constant, r_1, r_2 and r_3 decrease by a_1, a_2, a_3 respectively.

This is shown in the above table along with the effects on the residuals when y and z are given unit increments. It can be noted that the operation table consists of the unit matrix I and transpose of the co-efficient matrix.

At each step the numerically largest residual is reduced almost zero. To reduce a particular residual the value of corresponding variable is changed *i.e.* to reduce say r_2 by αy should be increased by α/b_2. When all the residuals have been reduced to almost zero, then the increments in x, y, z are added separately to give the desired solution.

Note. After finding x, y, z substitute them in (1), and check whether the residuals are negligible or not. If not then there is some mistake and the entire process should be rechecked.

3.29.3. (a) Convergency of the relaxation method

This method can be applied successfully only if the diagonal elements of the co-efficient matrix dominate the other coefficients in the corresponding row *i.e.*, if in the equations (1)

$|a_1| \geq |b_1| + |c_1|$; $|b_2| \geq |a_2| + |c_2|$; and $|c_3| \geq |a_3| + |b_3|$ with strict inequality for at least one row.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve by relaxation method, the equations*

$$10x - 2y - 2z = 6 \; ; -x + 10y - 2z = 7 \; ; -x - y + 10z = 8.$$

Sol. The residuals r_1, r_2, r_3 are given by

$$r_1 = 6 - 10x + 2y + 2z \; ; r_2 = 7 + x - 10y + 2z \; ; r_3 = 8 + x + y - 10z$$

The operation table is

x	y	z	r_1	r_2	r_3	
1	0	0	− 10	1	1	$\leftarrow L_1$
0	1	0	2	− 10	1	$\leftarrow L_2$
0	0	1	2	2	− 10	$\leftarrow L_3$

The relaxation table is

x	y	z	r_1	r_2	r_3	
0	0	0	6	7	8	$\leftarrow L_4$
0	0	1	8	9	– 2	$\leftarrow L_5 = L_4 + L_3$
0	1	0	10	– 1	– 1	$\leftarrow L_6 = L_5 + L_2$
1	0	0	0	0	0	$\leftarrow L_7 = L_6 + L_1$

Explanation. (*i*) In the line L_4 largest residual = 8

To reduce it we give an increment $\frac{8}{c_3} = \frac{8}{10} = 0.8 \simeq 1$

The resulting residuals are obtained by $L_4 + (1) L_3$ *i.e.*, line L_5.

(*ii*) In the line L_5 largest residual = 9

$\therefore$ Increment $= \frac{9}{b_2} = \frac{9}{10} = 0.9 \simeq 1$

The resulting residuals $(= L_6) = L_5 + 1.L_2$

(*iii*) In the line L_6 largest residual = 10

$\therefore$ Increment $= \frac{10}{a_1} = \frac{10}{10} = 1$

The resulting residuals $(= L_7) = L_6 + 1.L_1$, which are all zero's.

$\Rightarrow$ Exact solution is arrived and it is $x = 1, y = 1, z = 1$.

Example 2. *Solve by relaxation method, the equations*

$$9x - y + 2z = 9,\ x + 10y - 2z = 15,\ 2x - 2y - 13z = -17.$$

Sol. The residuals r_1, r_2, r_3 are given by

$$r_1 = 9 - 9x + y - 2z\ ;\ r_2 = 15 - x - 10y + 2z\ ;\ r_3 = -17 - 2x + 2y + 13z$$

The operation table is

x	y	z	r_1	r_2	r_3	
1	0	0	– 9	– 1	– 2	$\leftarrow L_1$
0	1	0	1	– 10	2	$\leftarrow L_2$
0	0	1	– 1	2	13	$\leftarrow L_3$

The relaxation table is

x	y	z	r_1	r_2	r_3	
0	0	0	9	15	– 17	$\leftarrow L_4$
0	0	1	7	17	– 4	$\leftarrow L_5 = L_4 + 1.L_3$
0	1	0	8	7	– 2	$\leftarrow L_6 = L_5 + 1.L_2$
0.89	0	0	– 0.01	6.11	– 3.78	$\leftarrow L_7 = L_6 + 0.89L_1$
0	0.61	0	0.6	0.01	– 2.56	$\leftarrow L_8 = L_7 + 0.61L_2$
0	0	0.19	0.22	0.39	– 0.09	$\leftarrow L_9 = L_8 + 0.1L_3$
0	0.039	0	0.259	0	– 0.012	$\leftarrow L_{10} = L_9 + 0.039L_2$
0.028	0	0	0.007	– 0.028	– 0.068	$\leftarrow L_{11} = L_{10} + 0.028L_1$
0	0	0.00523	– 0.00346	– 1.01754	– 0.00001	$\leftarrow L_{12} = L_{11} + 0.00523L_3$

Thus $x = 0.89 + 0.028 = 0.918$; $y = 1 + 0.61 + 0.039 = 1.649$ and $z = 1 + 0.19 + 0.00523 = 1.19523$.

Now substituting the values of x, y, z in (1), we get

$$r_1 = 9 - 9(0.918) + 1.649 - 2(1.19523) = -0.00346$$
$$r_2 = 15 - 0.918 - 10(1.649) + 2(1.19523) = -0.01754$$
$$r_3 = -17 - 2(0.918) + 2(1.649) + 13(1.19523) = -0.00001$$

which are in agreement with the final residuals in the table.

TEST YOUR KNOWLEDGE

Solve the following system of linear equations by (i) Gauss ellmination method (ii) Gauss Jordan method, (iii) Triangularization method and (iv) Crout's method :

1. $3x + 4y - z = 8$; $-2x + y + z = 3$; $x + 2y - z = 2$
2. $10x + y + z = 12$; $x + 10y + z = 12$; $x + y + 10z = 12$
3. $3x + y + 2z = 3$; $2x - 3y - z = -3$; $x - 2y + z = -4$
4. $2x - 6y + 8z = 24$; $5x + 4y - 3z = 2$; $3x + y + 2z = 16$
5. $x + 2y - z = 3$; $3x - y + 2z = 1$, $2x - 2y + 3z = 2$
6. $2x - 3y + z = -1$, $x + 4y + 5 = 25$, $3x - 4y + z = 2$
7. $2x + y + 4z = 12$, $8x - 3y + 2z = 20$; $4x + 11y - z = 33$
8. $2x + y + z = 10$, $3x + 2y + 3z = 18$, $x + 4y + 9z = 16$
9. $x + 2y + z = 8$; $2x + 3y + 4z = 20$, $4x + 3y + 2z = 16$
10. $5x_1 + 2x_2 + x_3 = 12$; $-x_1 + 4x_2 + 2x_3 = 2$, $2x_1 - 3x_2 + 10x_3 = -45$
11. $x_1 - x_2 - x_3 - x_4 = 2$; $2x_1 + 4x_2 - 3x_3 = 6$, $3x_2 - 4x_3 - 2x_4 = -1$; $-2x_1 + 4x_3 + 3x_4 = -3$
12. $x_1 + 2x_2 + 3x_3 + 4x_4 = 20$; $3x_1 - 2x_2 + 8x_3 + 4x_4 = 26$, $2x_1 + x_2 - 4x_3 + 7x_4 = 10$; $4x_1 + 2x_2 - 8x_3 - 4x_4 = 2$
13. $5x_1 + x_2 + x_3 + x_4 = 4$; $x_1 + 7x_2 + x_3 + x_4 = 12$, $x_1 + x_2 + 6x_3 + x_4 = -5$; $x_1 + x_2 + x_3 + 4x_4 = -6$
14. $2x_1 + x_2 + 5x_3 + x_4 = 5$; $x_1 + x_2 - 3x_3 + 4x_4 = -1$, $3x_1 + 6x_2 - 2x_3 + x_4 = 8$; $2x_1 + 2x_2 + 2x_3 - 3x_4 = 2$
15. $x + y + 2z - w = 5$; $x + 3y + 2z + w = 17$, $x + y + 3z + 2w = 20$; $x + 3y + 4z + 2w = 27$.

Solve the following system of linear equations (i) by Gauss-Jacobi's method and (ii) by Gauss-Seidel iteration method :

16. $2x + y + z = 4$, $x + 2y + z = 4$; $x + y + 2z = 4$
17. $8x + y + z = 8$; $2x + 4y + z = 4$; $x + 3y + 5z = 5$
18. $5x + 2y + z = 12$, $x + 4y + 2z = 15$, $x + 2y + 5z = 20$
19. $9x + 2y + 4z = 20$, $x + 10y + 4z = 6$, $2x - 4y + 10z = -15$
20. $54x + y + z = 110$, $2x + 15y + 6z = 72$, $-x + 6y + 27z = 85$
21. $28x - 4y - z = 32$, $x + 3y + 10z = 24$, $2x + 17y + 4z = 35$
22. $5x - y + z = 10$, $2x + 4y = 12$, $x + y + 5z = -1$
23. $10x_1 - 5x_2 - 2x_3 = 3$, $4x_1 - 10x_2 + 3x_3 = -3$, $x_1 + 6x_2 + 10x_3 = -3$
24. $10x + 2y + z = 9$, $2x + 20y - 2z = -44$, $-2x + 3y + 10z = 22$
25. $10x_1 + 7x_2 + 8x_3 + 7x_4 = 32$; $7x_1 + 5x_2 + 6x_3 + 5x_4 = 23$; $8x_1 + 6x_2 + 10x_3 + 9x_4 = 33$; $7x_1 + 5x_2 + 9x_3 + 10x_4 = 31$.

Solve by the relaxation method, the following equations :

26. $9x - 2y + z = 50$, $x + 5y - 3z = 18$, $-2x + 2y + 7z = 19$
27. $3x + 9y - 2z = 11$, $4x + 2y + 13z = 24$, $4x - 4y + 3z = -8$
28. $4.215x - 1.212y + 1.105z = 3.216$, $-2.120x + 3.505y - 1.632z = 1.247$, $1.122x - 1.313y + 3,986z = 2.112$
29. $10x - 2y - 3z = 305$, $-2x + 10y - 2z = 154$, $-2x - y + 10z = 120$
30. $8x_1 + x_2 + x_3 + x_4 = 14$; $2x_1 + 10x_2 + 3x_3 + x_4 = -8$, $x_1 - 2x_2 - 20x_3 + 3x_4 = 111$, $3x_1 + 2x_2 + 2x_3 + 19x_4 = 53$.

3.28.2. Gauss-Jordan Method

This method is a modified form of Gauss-elimination method. In this method the co-efficient matrix A of AX = B is reduced to a diagonal matrix or unit matrix by making all the elements above and below the principal diagonal of A as zeros. The labour of back substitution is saved here even though it involves additional computations.

Example. *Solve the following equations by Gauss-Jordan method*

$$x + 2y + z - w = -2 \; ; \; 2x + 3y - z + 2w = 7$$

$$x + y + 3z - 2w = -6 \; ; \; x + y + z + w = 2.$$

Sol. The given system in matrix form is

$$\underset{A}{\begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 3 & -1 & 2 \\ 1 & 1 & 3 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}} = \underset{B}{\begin{bmatrix} -2 \\ 7 \\ -6 \\ 2 \end{bmatrix}}$$

The argumented matrix is $[A \mid B] = \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 2 & 3 & -1 & 2 & 7 \\ 1 & 1 & 3 & -2 & -6 \\ 1 & 1 & 1 & 1 & 2 \end{array}\right]$

$$\begin{array}{l} R_2 \to -(R_2 - 2R_1) \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & -2 \\ 0 & 1 & 3 & -4 & -11 \\ 0 & -1 & 2 & -1 & -4 \\ 0 & -1 & 0 & 2 & 4 \end{array}\right]$$

$$\begin{array}{l} R_1 \to R_1 - 2R_1 \\ R_3 \to \frac{1}{5}(R_3 + R_2) \\ R_4 \to R_4 + R_2 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & -5 & 7 & 20 \\ 0 & 1 & 3 & -4 & -11 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 3 & -2 & -7 \end{array}\right]$$

$$\begin{array}{l} R_1 \to R_1 + 5R_3 \\ R_2 \to R_2 - 3R_3 \\ R_4 \to R_4 - 3R_3 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{array}{l} R_1 \to R_1 - 2R_4 \\ R_2 \to R_2 + R_4 \\ R_3 \to R_3 + R_4 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$\therefore$ The system AX = B reduces to the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

i.e., $x = 1, y = 0, z = -1$ and $w = 2$.

3.28.3. Method of Factorisation or Triangularisation

Consider the system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

These equations can be written in matrix form as AX = B ...(1) where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

In this method we use the fact that the square matrix A can be factorized into the form LU, where L being a unit lower triangular matrix and U being an upper triangular matrix iff all the minors of A are non-singular.

Let $A = LU$ ...(2)

$$\text{where } L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Now the equation (1) becomes LUX = B ...(3)

Setting UX = Y ...(4) then (3) $\Rightarrow$ LY = B ...(5)

i.e.,

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

giving $y_1 = b_1$, $l_{21}y_1 + y_2 = b_2$ and $l_{31}y_1 + l_{32}y_2 + y_3 = b_3$.

From these y_1, y_2, y_3 can be solved by forward substitution. Now from (4), we have

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

i.e., $u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1$; $u_{22}x_2 + u_{23}x_3 = y_2$ and $u_{33}x_3 = y_3$. These results x_1, x_2, x_3 by back substitution.

Now L and U can be found from LU = A

i.e.,
$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Equating the corresponding co-efficients, we get

$$u_{11} = a_{11},\ u_{12} = a_{12},\ u_{13} = a_{13}$$

$$l_{21}u_{11} = a_{21} \quad \text{or} \quad l_{21} = \frac{a_{21}}{a_{11}} \ ; \quad l_{31}u_{11} = a_{31} \quad \text{or} \quad l_{31} = \frac{a_{21}}{a_{11}}$$

$$l_{21}u_{12} + u_{22} = a_{22} \quad \text{or} \quad u_{22} = \frac{1}{a_{11}}(a_{11}a_{22} - a_{21}a_{12})$$

$$l_{21}u_{13} + u_{23} = a_{23} \quad \text{or} \quad u_{23} = \frac{1}{a_{11}}(a_{11}a_{23} - a_{21}a_{13})$$

$$l_{31}u_{12} + l_{32}u_{22} = a_{32} \quad \text{or} \quad l_{32} = \frac{1}{u_{22}}(a_{32} - l_{31}u_{12})$$

and
$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33} \quad \text{or} \quad u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$$

$\therefore$ L and U are known.

Example. *Solve the following system, by the method of triangularisation :*

$2x - 3y + 10z = 3,\ -x + 4y + 2z = 20,\ 5x + 2y + z = -12.$

Sol. The given system is AX = B, where

$$A = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

Let LU = A, where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

$$\therefore \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}$$

$$\Rightarrow \quad u_{11} = 2,\ u_{12} = -3,\ u_{13} = 10$$

$$l_{21}u_{11} = -1 \qquad \therefore \quad l_{21} = -\frac{1}{2} \ ;\ l_{31}u_{11} = 5 \quad \therefore \quad l_{31} = \frac{5}{2}$$

$$l_{21}u_{12} + u_{22} = 4 \qquad \therefore \quad u_{22} = 4 - l_{21}u_{12} = 4 - \left(-\frac{1}{2}\right)(-3) = \frac{5}{2}$$

$$l_{21}u_{13} + u_{23} = 2 \qquad \therefore \quad u_{23} = 2 - l_{21}u_{13} = 2 - \left(-\frac{1}{2}\right)(10) = 7$$

$$l_{31}u_{12} + l_{32}u_{22} = 5 \qquad \therefore \quad l_{32} = \frac{1}{u_{22}}[5 - l_{31}u_{12}] = \frac{19}{5}$$

and $$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1 \qquad \therefore \quad u_{33} = 1 - l_{31}u_{13} - l_{32}u_{23} = -\frac{253}{5}.$$

$$\therefore \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{5}{2} & \frac{19}{5} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & -3 & 10 \\ 0 & \frac{5}{2} & 7 \\ 0 & 0 & -\frac{253}{5} \end{bmatrix}$$

Let $UX = Y$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, then $LY = B$ *i.e.*,

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{5}{2} & \frac{19}{5} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix} \qquad \ldots(1)$$

and $$\begin{bmatrix} 2 & -3 & 10 \\ 0 & \frac{5}{2} & 7 \\ 0 & 0 & -\frac{253}{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad \ldots(2)$$

Now (1) $\Rightarrow$ $$y_1 = 3 \,;\, -\frac{1}{2}y_1 + y_2 = 20 \,;\, \frac{5}{2}y_1 + \frac{19}{5}y_2 + y_3 = -12$$

$$\therefore \qquad y_1 = 3, y_2 = \frac{43}{2}, y_3 = -\frac{506}{5}.$$

From (2), we have (after replacing y_1, y_2 and y_3 from above)

$$2x - 3y + 10z = 3 \,;\, \frac{5}{2}y + 7z = \frac{43}{2} \,;\, -\frac{253}{5}z = \frac{-506}{5}$$

Solving these, we get by back substitution $x = -4, y = 3$ and $z = 2$.

3.28.4. Crout's Methods

This method is superior to the Gauss elimination method because it requires less calculation. It is based on the fact that every square matrix A can be expressed as the product of a lower triangular matrix and a unit upper triangular matrix. Let AX = B ...(1) be the given system and let A = LU ...(2) where

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Then (1) becomes $LUX = B$ ...(3) Let $UX = Y$ ...(4)

so that (3) becomes $LY = B$ ...(5)

i.e.,
$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\therefore \quad l_{11}y_1 = b_1 \; ; \; l_{21}y_1 + l_{22}y_2 = b_2 \; ; \; l_{31}y_1 + l_{32}y_2 + l_{33}y_3 = b_3$$

By forward substitution y_1, y_2, y_3 can be found out if L is known. From (4)

$$\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \text{...(6)}$$

i.e.,
$$x_1 + u_{12}x_2 + u_{13}x_3 = y_1 \; ; \; x_2 + u_{23}x_3 = y_2 \; ; \; x_3 = y_3$$

By back substitution x_1, x_2, x_3 can be found out if U is known.

Now L and U can be found from $LU = A$.

i.e.,
$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

or
$$\begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Equating the corresponding elements on both sides, we get

$$l_{11} = a_{11}, l_{21} = a_{21}, l_{31} = a_{31}$$

$$l_{11}u_{12} = a_{12} \quad \therefore \quad u_{12} = \frac{a_{12}}{a_{11}}$$

$$l_{11}u_{13} = a_{13} \quad \therefore \quad u_{13} = \frac{a_{13}}{a_{11}}$$

$$l_{21}u_{12} + l_{22} = a_{22} \quad \therefore \quad l_{22} = a_{22} - l_{21}u_{12}$$

$$l_{31}u_{12} + l_{32} = a_{32} \quad \therefore \quad l_{32} = a_{32} - l_{31}u_{12}$$

$$l_{21}u_{13} + l_{22}u_{23} = a_{23} \quad \therefore \quad u_{23} = \frac{1}{u_{23}}\{a_{23} - l_{21}u_{13}\}$$

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = a_{33} \quad \therefore \quad l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$$

Thus all the 12 unknowns are determined and the solution, as shown already, is got from (6).

Crout during the above decomposition of the co-efficient matrix A, devised a technique which is given here under to determine 12 unknowns systematically.

3.28.4. (a) Computation Scheme by Crout's Method

The argumented matrix of the system $AX = B$ ...(1) is

$$[A \mid B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

The matrix of 12 unknowns so called derived matrix or auxiliary matrix is

$$\begin{bmatrix} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{bmatrix}$$

and is to be calculated as follows :

Step 1. The **first column** of the derived matrix (D.M) is identical with the first column of [A | B].

Step 2. The **first row** to the right of the first column of the D.M is got by dividing the corresponding element in [A | B] by the leading diagonal element of that row.

Step 3. Remaining second column of D.M

$$\because \quad l_{22} = a_{22} - l_{21}u_{12} \,;\, l_{32} = a_{32} - l_{31}u_{12}$$

$$\left.\begin{matrix}\text{Each element on}\\ \text{or below the diagonal}\end{matrix}\right\} = \left\{\begin{matrix}\text{Corresponding element in } [A\,|\,B] - \text{The product}\\ \text{of the first element in that row and in that column.}\end{matrix}\right.$$

Step 4. Remaining elements of second row of D.M

$$\text{Each element} = \left\{\begin{matrix}[\text{Corresponding element in } [A\,|\,B]\\ - \text{The product of the first element in that row and in}\\ \text{that column}] \div [\text{leading diagonal element in that row}]\end{matrix}\right.$$

i.e.,

$$u_{23} = \frac{a_{23} - l_{23}u_{13}}{l_{22}} \,;\, y_2 = \frac{b_2 - l_{21}y_1}{l_{32}}.$$

Step 5. Remaining elements of third column of D.M

$$\text{Each element} = \left\{\begin{matrix}\text{Corresponding element of } [A \mid B]\\ - \text{Sum of the inner products of the previously}\\ \text{calculated elements in the same row and column}\end{matrix}\right.$$

i.e.,

$$l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$$

Step 6. Remaining elements of third row of D.M

$$\text{Each element} = \left\{\begin{matrix}[\text{Corresponding element of } [A\,|\,B]\\ - \text{Sum of the inner products of the previously}\\ \text{calculated elements in the same row and column}]\\ \div [\text{the leading diagonal element in that row}]\end{matrix}\right.$$

i.e.,

$$y_3 = \frac{b_3 - (l_{31}y_1 + l_{32}y_2)}{l_{33}}.$$

Now the matrices L, U and Y can be written and hence X from UX = Y.

Note. The above procedure holds good for any order square matrix A.

Example. *Solve the system*

$2x + y + 4z = 12$, $8x - 3y + 2z = 20$, $4x + 11y - z = 33$ by Crout's method.

Sol. Argumented matrix = [A | B] = $\begin{bmatrix} 2 & 1 & 4 & 12 \\ 8 & -3 & 2 & 10 \\ 4 & 11 & -1 & 23 \end{bmatrix}$

Let the derived matrix (D.M) = $\begin{bmatrix} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{bmatrix}$

(1) Elements of the first column of D.M are $l_{11} = 2, l_{21} = 8, l_{31} = 4$.

(2) Elements of the first row to the right of the first column

$$u_{12} = \frac{a_{12}}{l_{11}} = \frac{1}{2}\,;\; u_{13} = \frac{a_{13}}{l_{11}} = \frac{4}{2} = 2\,;\; y_1 = \frac{b_1}{l_{11}} = \frac{12}{2} = 6.$$

(3) Elements of the remaining second column

$$l_{22} = a_{22} - u_{12}l_{21} = (-3) - \left(\frac{1}{2}\right)(8) = -7$$

$$l_{32} = a_{32} - u_{12}l_{31} = 11 - \left(\frac{1}{2}\right)(4) = 9$$

(4) Elements of the remaining second row

$$u_{23} = \frac{a_{23} - u_{13}l_{21}}{l_{22}} = -\frac{1}{7}\{2 - (2)(8)\} = 2$$

$$y_2 = \frac{b_2 - l_{21}y_1}{l_{22}} = \frac{20 - (8)(6)}{-7} = 4.$$

(5) Remaining third column $l_{33} = a_{33} - (l_{31}u_{13} + l_{32}u_{23}) = -1 - (4)(2) - (9)(2) = -27$.

(6) Remaining third row $\quad y_3 = \dfrac{b_3 - (l_{31}y_1 + l_{32}y_2)}{l_{33}} = \dfrac{33 - (4)(6) - (9)(4)}{-27} = 1$

$$\therefore \qquad \text{D.M} = \begin{bmatrix} 2 & 1/2 & 2 & 6 \\ 8 & -7 & 2 & 4 \\ 4 & 9 & -27 & 1 \end{bmatrix}$$

Now the solution is got from the system UX = Y

i.e.,
$$\begin{bmatrix} 1 & 1/2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

which is equivalent to $x + \frac{1}{2}y + 2z = 6\,;\, y + z = 4,\, z = 1$

By back substitution $x = 3, y = 2$ and $z = 1$.